DETERMINATION OF THE THERMAL BOUNDARY CONDITIONS FROM NONSTATIONARY-TEMPERATURE MEASUREMENT DATA

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A method of constructing a solution of the converse heat-conduction problem is presented. The inverse operator is represented in explicit form with a regularization parameter which depends on the level of the error in the initial data.

In the majority of papers on the solution of ill-posed converse problems in heat conduction, the problem is reduced to setting up constructive (numerically realizable) methods of solving integral equations. However, these problems do not enable one to obtain the inverse operator in explicit form and lead to the solution of either linear algebraic systems or to finding the eigen vectors and numbers of matrices [1-3]. Below we describe a method of constructing the regularized inverse operator in explicit form, which enables one to obtain a constructive solution of the initial ill-posed problem as it applies to reconstructing the thermal boundary conditions from data obtained from measurements of the nonstationary temperatures in the constructional elements at a certain distance from the heated surface.

Suppose that in a heat experiment a model of a plane unbounded plate is set up, which is subjected to heating or cooling on one side and is thermally insulated on the other. From measurements of the temperature at different times at a point with coordinate $x = x_1$ it is required to determine the heat flux and the temperature on the surface $(x = \delta)$ subjected to heating or cooling. The thermal properties over the temperature range employed in the experiment are assumed to be constant. The mathematical formulation of the problem therefore has the following form:

$$\frac{\partial^2 T(\mathrm{Fo},\ \xi)}{\partial \xi^2} = \frac{\partial T(\mathrm{Fo},\ \xi)}{\partial \mathrm{Fo}}, \tag{1}$$

$$T(0,\ \xi) = 0;\quad \left.\frac{\partial T(\mathrm{Fo},\ \xi)}{\partial \xi}\right|_{\xi=0} = 0;\quad T(\mathrm{Fo},\ \xi)|_{\xi=\xi_1} = T(\mathrm{Fo},\ \xi_1);$$

$$\left.\frac{\partial T(\mathrm{Fo},\ \xi)}{\partial \xi}\right|_{\xi=1} = ?,\quad T(\mathrm{Fo},\ 1) = ? \tag{1'}$$

The solution of problem (1) in Laplace-transform space can be written in the following form [4]:

$$\bar{q}(s,\ 1) = \frac{T(s,\ \xi_1)\sqrt{s}\,\mathrm{sh}\sqrt{s}}{\mathrm{ch}\sqrt{s}\,\xi_1};\quad T(s,\ 1) = \frac{T(s,\ \xi_1)\,\mathrm{ch}\sqrt{s}}{\mathrm{ch}\sqrt{s}\,\xi_1}. \tag{2}$$

The ill-posed nature of the problem in this case manifests itself in the fact that the transforms $\sqrt{s}\mathrm{sh}\sqrt{s}/\mathrm{ch}\sqrt{s}\xi_1$ and $\mathrm{ch}\sqrt{s}/\mathrm{ch}\sqrt{s}\xi_1$ with $\xi < 1$, due to the fact that they approach ∞ as $|s| \to \infty$, do not satisfy one of the transformation requirements and do not have originals in the form of ordinary functions. To obtain a solution which can be transformed we will use Eq. (1) and introduce into it a "regularizing source" which depends on T(Fo, ξ) and the parameter β

$$\frac{\partial^2 T(\mathrm{Fo},\ \xi)}{\partial \xi^2} + \int_0^{\mathrm{Fo}} \frac{\partial^2 T(\tilde{\mathrm{Fo}},\ \xi)}{\partial \tilde{\mathrm{Fo}}^2} \exp\left[-\beta(\mathrm{Fo} - \tilde{\mathrm{Fo}})\right] d\tilde{\mathrm{Fo}} = \frac{\partial T(\mathrm{Fo},\ \xi)}{\partial \mathrm{Fo}}. \tag{3}$$

It is obvious that $\beta\exp[-\beta(\mathrm{Fo} - \tilde{\mathrm{Fo}})] \to \delta(\mathrm{Fo} - \tilde{\mathrm{Fo}})$ when $\beta \to \infty$, where $\delta(\mathrm{Fo} - \tilde{\mathrm{Fo}})$ is the Dirac function, and if the term introduced into (3) has the form

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$$\int_0^{\mathrm{Fo}} \frac{\partial^2 T(\tilde{\mathrm{Fo}}, \xi)}{\partial \tilde{\mathrm{Fo}}^2} \frac{\beta}{\beta_0} \exp[-\beta(\mathrm{Fo} - \tilde{\mathrm{Fo}})]\, d\tilde{\mathrm{Fo}} \underset{\beta\to\infty}{\to} \frac{1}{\beta_0} \frac{\partial^2 T(\mathrm{Fo}, \xi)}{\partial \mathrm{Fo}^2}, \tag{4}$$

for fairly large β_0 the problem would become similar to the quasitransformation method [5]. In this case $\beta = \beta_0$ and when $\beta_0 \to \infty$ the last term in (3) would approach zero. We apply a Laplace transformation to (3)

$$sT(s, \xi) - \frac{s^2 T(s, \xi)}{s+\beta} = \frac{d^2 T(s, \xi)}{d\xi^2} \tag{5}$$

when $T(0, \xi) = \left.\frac{\partial T(\mathrm{Fo}, \xi)}{\partial \mathrm{Fo}}\right|_{\mathrm{Fo}=0} = 0$. The solution of (5), taking (1') into account, has the form

$$\bar{q}(s, 1) = T(s, \xi_1) \sqrt{\frac{s\beta}{s+\beta}} \frac{\mathrm{sh}\sqrt{\frac{s\beta}{s+\beta}}}{\mathrm{ch}\sqrt{\frac{s\beta}{s+\beta}}\,\xi_1}, \tag{6}$$

$$T(s, 1) = T(s, \xi_1) \frac{\mathrm{ch}\sqrt{\frac{s\beta}{s+\beta}}}{\mathrm{ch}\sqrt{\frac{s\beta}{s+\beta}}\,\xi_1}. \tag{7}$$

Henceforth we will confine ourselves to obtaining a solution for $\bar{q}(\mathrm{Fo}, 1)$. Expression (6) can be converted as follows:

$$\bar{q}(s, 1) = T(s, \xi_1) \sqrt{\frac{s\beta}{s+\beta}} \left\{ \mathrm{sh}\sqrt{\frac{s\beta}{s+\beta}}(1-\xi_1) + \mathrm{th}\sqrt{\frac{s\beta}{s+\beta}}\,\xi_1\, \mathrm{ch}\sqrt{\frac{s\beta}{s+\beta}}(1-\xi_1) \right\}. \tag{8}$$

It is obvious that

$$\lim_{\beta\to\infty} \bar{q}(s, 1) = T(s, \xi_1)\sqrt{s}\{\mathrm{sh}\sqrt{s}(1-\xi_1) + \mathrm{th}\sqrt{s}\,\xi_1\, \mathrm{ch}\sqrt{s}(1-\xi_1)\},$$

which corresponds to the relationship between $\bar{q}(\mathrm{Fo}, 1)$ and $T(\mathrm{Fo}, \xi_1)$ in transformation space when solving the initial system (1). However, whereas (2) cannot be transformed in the sense of obtaining the inverse operator that is bounded when $\mathrm{Fo} = 0$, Eq. (8) can be so transformed. In this case

$$\bar{q}(\mathrm{Fo}, 1) = \int_0^{\mathrm{Fo}} \frac{dT(\tilde{\mathrm{Fo}}, \xi_1)}{d\tilde{\mathrm{Fo}}} \Psi(\mathrm{Fo} - \tilde{\mathrm{Fo}})\, d\tilde{\mathrm{Fo}}. \tag{9}$$

The function Ψ is determined by the values for $\mathrm{Fo} - \tilde{\mathrm{Fo}}$ of the original of the expression

$$\Psi(s) = \frac{1}{s} \sqrt{\frac{s\beta}{s+\beta}} \left\{ \mathrm{sh}\sqrt{\frac{s\beta}{s+\beta}}(1-\xi_1) + \mathrm{th}\sqrt{\frac{s\beta}{s+\beta}}\,\xi_1\, \mathrm{ch}\sqrt{\frac{s\beta}{s+\beta}}(1-\xi_1) \right\}. \tag{10}$$

The original of the first term in (10) can be obtained by expanding $\mathrm{sh}\sqrt{\frac{s\beta}{s+\beta}}(1-\xi_1)$ in series

$$\varphi_1(\theta) = \sum_{n=0}^{\infty} \frac{\beta^{n+1}(1-\xi_1)^{2n+1} L_n(\beta\theta)}{(2n+1)!} \exp(-\beta\theta), \tag{11}$$

where L_n are Laguerre polynomials of order n. To obtain the original of the second term we expand the hyperbolic tangent and cosine in series as in [6]

$$\varphi_2(s) = 2\xi_1 \sum_{k=1}^{\infty} \left[\xi_1^2 + \frac{(2k-1)^2\pi^2}{4\beta} \right]^{-1} \left\{ s + \frac{(2k-1)^2\pi^2}{4\left[\xi_1^2 + \frac{(2k-1)^2\pi^2}{4\beta}\right]} \right\}^{-1} \sum_{n=0}^{\infty} \frac{\beta^n (1-\xi_1)^{2n} s^n}{(2n)!(s+\beta)^n}. \tag{12}$$

We will put $(2k-1)^2\pi^2 / \left[4\xi_1^2 + \frac{(2k-1)^2\pi^2}{\beta}\right] = \gamma_k$. Then the expression $1/(s+\gamma_k)$ can be represented in the form $[1 + (\beta - \gamma_k)/(s+\gamma_k)]/(s+\beta)$. Substituting the result into (12) we obtain

$$\varphi_2(s) = 2\xi_1 \sum_{k=1}^{\infty} \left[1 + \frac{\beta-\gamma_k}{s+\gamma_k}\right] \Big/ \left[\xi_1^2 + \frac{(2k-1)^2\pi^2}{4\beta}\right] \sum_{n=0}^{\infty} \frac{\beta^n (1-\xi_1)^{2n} s^n}{(2n)!(s+\beta)^{n+1}}. \tag{13}$$

Assuming that $2\xi_1 \sum_{k=1}^{\infty} (4\beta/\pi^2)/[(2k-1)^2 + \xi_1^2 4\beta/\pi^2] = \operatorname{th}\sqrt{\beta}\,\xi_1$, after changing into the space of the originals we arrive at the expression

$$\varphi_2(\theta) = \operatorname{th}\sqrt{\beta}\,\xi_1 \sum_{n=0}^{\infty} \frac{L_n(\beta\theta)\,\beta^n (1-\xi_1)^{2n}}{(2n)!} \exp(-\beta\theta) +$$
$$+ 2\xi_1 \int_0^{\theta} \sum_{k=1}^{\infty} \frac{\beta-\gamma_k}{\xi_1^2 + \frac{(2k-1)^2\pi^2}{4\beta}} \exp(-\gamma_k\theta') \sum_{n=0}^{\infty} \frac{\beta^n (1-\xi_1)^{2n} L_n[\beta(\theta-\theta')]}{(2n)!} \times \tag{14}$$
$$\times \exp[-\beta(\theta-\theta')]\, d\theta'.$$

Combining (11) and (14) we obtain

$$\Psi(\mathrm{Fo}-\tilde{\mathrm{Fo}}) = \sum_{n=0}^{\infty} \frac{\beta^{n+1}(1-\xi_1)^{2n+1} L_n[\beta(\mathrm{Fo}-\tilde{\mathrm{Fo}})]}{(2n+1)!} \exp[-\beta(\mathrm{Fo}-\tilde{\mathrm{Fo}})] +$$
$$+ \operatorname{th}\sqrt{\beta}\,\xi_1 \sum_{n=0}^{\infty} \frac{\beta^n (1-\xi_1)^{2n} L_n[\beta(\mathrm{Fo}-\tilde{\mathrm{Fo}})]}{(2n)!} \exp[-\beta(\mathrm{Fo}-\tilde{\mathrm{Fo}})] + \tag{15}$$
$$+ 2\xi_1 \int_0^{\mathrm{Fo}-\tilde{\mathrm{Fo}}} \sum_{k=1}^{\infty} \frac{(\beta-\gamma_k)}{\xi_1^2 + \frac{(2k-1)^2\pi^2}{4\beta}} \exp(-\gamma_k\theta) \sum_{n=0}^{\infty} \frac{\beta^n (1-\xi_1)^{2n} L_n[\beta(\mathrm{Fo}-\tilde{\mathrm{Fo}}-\theta)]}{(2n)!} \exp[-\beta(\mathrm{Fo}-\tilde{\mathrm{Fo}}-\theta)]\, d\theta.$$

The possibility of obtaining a realizable solution of the form (9) numerically sufficiently close to the desired solution $\tilde{q}_d$ is obviously determined by the behavior of the function Ψ for large values of β. In fact, it is easy to show that the solution of the direct problem (4) reduces to the solution of the direct initial problem (1) as $\beta \to \infty$. On the other hand, Eq. (15) as $\beta \to \infty$ has no limit in the usual sense, and reduces to the sum of δ-functions and its derivatives. Convolution of the latter with T'(Fo) gives a solution of the form [7] which is the sum of the m-th derivatives of the experimental function. As is well known, m-fold differentiation of the experimental function is generally an ill-posed operation and such a solution is therefore not constructive. Since the initial data are always known with a certain error, to construct an approximate solution it is best to calculate (15) not for arbitrarily large β, but for values which are matched to the level of the error in the initial data. In fact, it follows from an analysis of (15) that $\Psi(\mathrm{Fo}-\tilde{\mathrm{Fo}})$ for large β contains rapidly oscillating functions with a fairly large weight, the presence of which, when (9) is evaluated numerically, leads to large errors. To choose the optimum value of β we will consider the difference

$$\Delta\varphi(s) = T(s,\xi_1) - T(s,\xi_1) \times$$
$$\times \frac{\sqrt{\frac{s\beta}{s+\beta}} \left\{ \operatorname{sh}\sqrt{\frac{s\beta}{s+\beta}}(1-\xi_1) + \operatorname{th}\sqrt{\frac{s\beta}{s+\beta}}\,\xi_1 \operatorname{ch}\sqrt{\frac{s\beta}{s+\beta}}(1-\xi_1) \right\}}{\sqrt{s}\operatorname{sh}\sqrt{s}}. \tag{16}$$

The last term in (16) obviously represents a solution in transforms of the accurate direct problem with thermal flux calculated from (9). The original of $\Delta\varphi(\mathrm{Fo})$ has the form

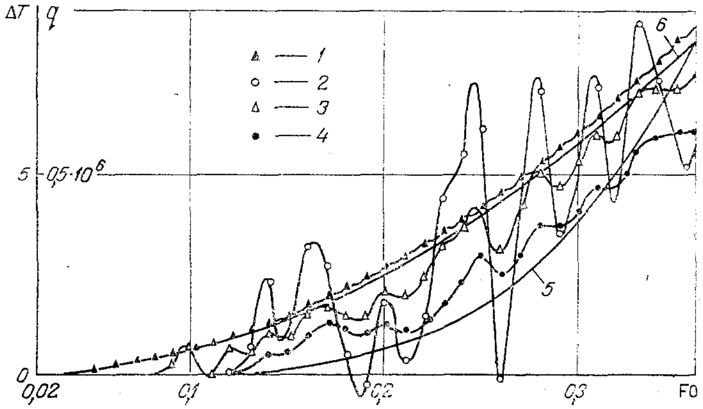

Fig. 1. Results of a calculation of the heat flux q(Fo) on the surface: 1) $\beta = 90$, $\sigma = 0$; 2) 40, 0.01; 3) 40, 0.05; 4) 20, 0.05; 5) the true values of the temperature of the thermally insulated wall; and 6) the heat flux. q, W/m^2, ΔT, °K.

$$\Delta\varphi(\mathrm{Fo}) = \int_0^{\mathrm{Fo}} \frac{dT(\tilde{\mathrm{Fo}}, \xi_1)}{d\tilde{\mathrm{Fo}}} [1 - \Psi_1(\beta, \mathrm{Fo} - \tilde{\mathrm{Fo}})]\, d\tilde{\mathrm{Fo}}. \tag{17}$$

The function $\Delta\varphi(\mathrm{Fo})$ therefore represents the difference between the measured temperature and the temperature calculated from the reconstructed heat flow. Hence, it is natural to impose the following condition on the choice of the parameter β:

$$\int_0^{\mathrm{Fo}} [\Delta\varphi(\mathrm{Fo})]^2 d\,\mathrm{Fo} \leqslant M \int_0^{\mathrm{Fo}^*} \varepsilon^2(\mathrm{Fo})\, d\,\mathrm{Fo}, \tag{18}$$

where Fo* is the length of the sample.

Below we give the results of a calculation of the specific heat flux on the surface of a plane wall from the known temperature on the thermally insulated side. We used, as accurate values of the temperatures, results obtained by solving the direct problem with a thermal flux varying as $q = 10^6 \mathrm{Fo}^2$ W/m^2, for a wall 10 mm thick with $a = 0.04$ m^2/h and $\lambda = 40$ W/m·K. The figure shows theoretical values of the heat fluxes calculated from (9) for different β (from 20 to 90). To explain the effect of the random errors in the initial data, the results of a calculation of the direct problem (the "accurate" values) were distorted by introducing random numbers with a uniform distribution. One can easily follow from the graphs the effect of the parameter β and the error level, introduced into the "accurate" initial data, on the results of the calculation of q(Fo, 1). As can be seen from the figure, the regularizing effect of β manifests itself in an increase in the stability of the solution as β decreases when the error level in the initial data increases. Then, when the parameter β is reduced, the solution on the "accurate" initial data, as might have been expected, differs even more from the accurate solution. An obvious compromise is to choose the optimum value of the parameter β from Eq. (18).

NOTATION

T, temperature, °K; q, specific heat flux, W/m^2; $\bar{q}$, dimensionless thermal flux, $\bar{q} = q\delta/(\lambda T_0)$; x, x_1, coordinate, m; ξ, ξ_1, dimensionless coordinate, $\xi = x/\delta$; δ, plate thickness, m; a, thermal diffusivity, m^2/h; λ, thermal conductivity, W/m·K; Fo, $\tilde{\mathrm{Fo}}$, Fo*, Fourier numbers, $\mathrm{Fo} = a\tau/\delta^2$; τ, time, h; σ, mean-square error of the results of temperature measurements, °K; β, regularization parameter.

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EXPLICIT SOLUTIONS OF MULTIDIMENSIONAL INVERSE UNSTEADY HEAT-CONDUCTION PROBLEMS

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Explicit solutions are found of a number of inverse problems of determining the thermal conductivity in linear and nonlinear heat transport.

The determination of variable thermophysical characteristics of media is one of the urgent problems of contemporary thermophysics. Recently there has been a rapid development of the theory of multidimensional inverse problems [1-5]. In these investigations great importance is attached to the development of special methods which yield explicit solutions. These solutions can serve directly as a basis for experimental methods of determining variable physical characteristics of media.

We consider a thermal process described by the system

$$C(x,\ t)\,T_t-\nabla\lambda(x,\ t)\nabla T+\alpha(x,\ t)\,T=Q(x,\ y,\ t), \tag{1}$$

$$T|_{t=0}=\varphi(x,\ y), \tag{2}$$

$$T|_{\bar{D}_1 X\Gamma_2}=0,\ T|_{\Gamma_1 X\bar{D}_2}=f(\xi,\ y,\ t). \tag{3}$$

If the quantities C, λ, α, Q, φ, and f are known, system (1)-(3) can be used to calculate the temperature distribution T(x, y, t). Our primary problem is to determine the thermal conductivity $\lambda(x, t)$. To do this we supplement system (1)-(3) by the condition

$$\left.\frac{\partial T}{\partial \nu}\right|_{y=\eta}=\gamma(x,\ t), \tag{4}$$

which is the expression for the temperature gradient on the plane y = η, where η is a fixed point on the boundary Γ_2. The coefficient $\lambda(x, t)$ is sought in the class of continuous and positive functions.

Questions of the correctness of problems of the type (1)-(4) were studied in [4]. We consider cases for which the solutions can be found in explicit form.

We denote by $\omega(y)$ the normalized eigenfunction of the operator $-\Delta_y$ corresponding to the eigenvalue $\mu > 0$, i.e.,

$$-\Delta_y\omega(y)=\mu\omega(y),\ \omega(y)|_{\Gamma_2}=0,\ y\in D_2. \tag{5}$$

If m = 1, $\bar{D}_2 \equiv [0, 1]$, then $\omega(y) = \sin k\pi y$, $\mu = k^2\pi^2$, where k is a positive integer. It is not difficult to indicate the general form of the function $\omega(y)$ for a number of other domains also.

We consider a thermal process in which the following conditions are realized:

a) $Q(x, y, t) = Q_0(x, t)\omega(y)$, $\varphi(x, y) = \varphi_0(x)\omega(y)$, $f(\xi, y, t) = f_0 \times (\xi, t)\,\omega(y)$, where $Q_0(x, t)$, $\varphi_0(x)$, $f_0(\xi, t)$ are given functions;

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